Gröbner Bases For Feistel Designs With Application To GMiMC

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Motivation

- In past 10 years many novel symmetric designs for Homomorphic Encryption (HE), Multi-Party Computation (MPC) and Zero-Knowledge (ZK) have been introduced.
    - Native over prime field $\mathbb{F}_p$.
    - Multiplicative complexity one of the main performance metrics.
    - Often low degree polynomials at round level.
    - Compiled list of Arithmetization-Oriented (AO) Primitives: `https://stap-zoo.com/all-stap-primitives/`.
- Algebraic attacks major challenge in cryptanalysis.
    - Interpolation attacks.
    - Polynomial system solving.

Feistel Networks

- Long track record in the literature.
- Well-understood statistical properties.
- Very flexible.

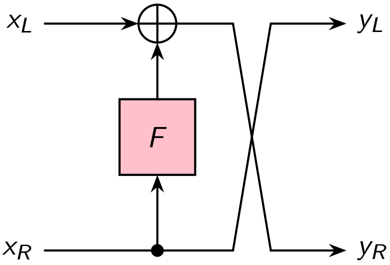

Figure: Classic two branch Feistel Network.

Arithmetization-Oriented Feistel Networks

Feistel Designs

- GMiMC (MPC, ZK) [AGP+19a].
- Ciminion (MPC) [GØSW23].
- Rubato (HE) [HKL+22].

Designs With Feistel Components

- Hydra (MPC) [GØSW23].
- Anemoi (ZK) [BBC+23].
- Monolith (ZK) [GKL+24].

Gröbner Bases I

- Invented by Austrian mathematician Bruno Buchberger in his PhD thesis at Universität Innsbruck [Buc65].
- Provides a general framework to solve fully determined polynomial systems.
- **Terminology:**
  - K a field, $\mathbb{F}_q$ a field with q elements.
  - $P = K[x_1, \ldots, x_n]$.
  - $I \subset P$ is an ideal if:
    - $0 \in I$.
    - $a, b \in I \Rightarrow a - b \in I$.
    - $r \in P,\ a \in I \Rightarrow a \cdot r \in I$.
  - We think of a polynomial system $\mathcal{F} \subset P$ as ideal $(\mathcal{F})$.

Gröbner Bases II

Term Orders on P

- A monomial $m = \prod_{i=1}^{n} x_i^{a_i} \in P$ can be identified with a vector $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^{n}$.
- A term order $>$ on P satisfies:
  - $>$ is a total ordering on $\mathbb{Z}_{\geq 0}^{n}$.
  - $\boldsymbol{a} > \boldsymbol{b}$ and $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}$, then $\boldsymbol{a} + \boldsymbol{c} > \boldsymbol{b} + \boldsymbol{c}$.
  - $>$ is a well-ordering on $\mathbb{Z}_{\geq 0}^{n}$.
- $f \in P$, then $\mathrm{LM}_{>}(f)$ is the largest monomial in f under $>$.

Gröbner Bases III

Degree Reverse Lexicographic (DRL) Term Order

$\boldsymbol{a} >_{DRL} \boldsymbol{b}$ if either

- $\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i$, or
- $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and right-most non-zero entry of $\boldsymbol{a} - \boldsymbol{b}$ is negative.

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Gröbner Basis

- $I \subset P$ ideal, $>$ term order on P.
- $\mathcal{G} \subset P$ is Gröbner basis if
  - $(\mathcal{G}) = I$, and
  - $(\mathrm{LM}_>(g) \mid g \in \mathcal{G}) = (\mathrm{LM}_>(f) \mid f \in I)$.

Gröbner Bases IV

Pairwise Coprime Leading Monomials

- $\mathcal{G} \subset P$, $>$ term order on P.
- $\forall f, g \in \mathcal{G}$, $f \neq g$,

$$\operatorname{gcd}\Big(\mathrm{LM}_>(f), \mathrm{LM}_>(g)\Big) = 1.$$

- $\Rightarrow \mathcal{G}$ is $>$-Gröbner basis [CLO15, Chapter 2 §9].

Gröbner Bases IV

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Example

- $\mathcal{F} = \{f_1, \ldots, f_n\} \subset K[x_1, \ldots, x_n]$ such that for all $1 \leq i \leq n$

$$\text{LM}_{DRL}(f_i) = x_i^d.$$

- $\Rightarrow \mathcal{F}$ is DRL Gröbner basis.

$\texttt{GMiMC}_{\text{erf}}$ I

- Operates on $\mathbb{F}_q^n$.
- Cubing as non-linear function.
- $c_i \in \mathbb{F}_q$ round constant.
- Two key sizes: $\boldsymbol{k} \in \mathbb{F}_q^n$ or $k \in \mathbb{F}_q$.
- Sponge mode for hash function.

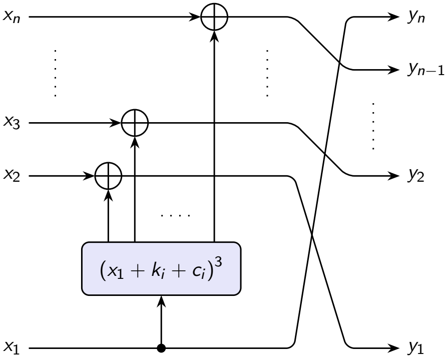

Figure: Keyed $\texttt{GMiMC}_{\text{erf}}$ round function.

$\texttt{GMiMC}_{\text{erf}}$ II

- Multivariate key $\boldsymbol{k} \in \mathbb{F}_q^n$.
  - Needs key schedule matrix $\boldsymbol{M}_{\mathcal{K}} \in \mathbb{F}_q^{n \times n}$ such that
    - $\det(\boldsymbol{M}_{\mathcal{K}}) \neq 0$, and
    - for all $1 \leq i \leq \left\lceil \frac{r}{n} \right\rceil$ all entries in $\boldsymbol{M}_{\mathcal{K}}^i$ are non-zero.
  - Set $\boldsymbol{K} = \left(\boldsymbol{M}_{\mathcal{K}} \boldsymbol{k}, \dots, \boldsymbol{M}_{\mathcal{K}}^{\left\lceil \frac{r}{n} \right\rceil} \boldsymbol{k} \right)^{\intercal} \in \mathbb{F}_q^{n \cdot \left\lceil \frac{r}{n} \right\rceil}$.
  - $k_i = K_i$.

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    - $k_i = K_i$.
- Univariate key $k \in \mathbb{F}_q$.
    - $k_i = k$ for all rounds.

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$\texttt{GMiMC}_{\mathsf{erf}}$ Polynomial System

- $\boldsymbol{p}, \boldsymbol{c} \in \mathbb{F}_q^n$ encrypted under key $\boldsymbol{k} \in \mathbb{F}_q^n$.

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- $\boldsymbol{p}, \boldsymbol{c} \in \mathbb{F}_q^n$ encrypted under key $\boldsymbol{k} \in \mathbb{F}_q^n$.
- Introduce variables after every round function

$$\mathcal{F}_{\texttt{GMiMC}_{\text{erf}}} : \begin{cases} \mathcal{R}^{(1)} \left(\boldsymbol{p}, \boldsymbol{y} \right) - \boldsymbol{x}^{(1)} = 0, & \\ \mathcal{R}^{(i)} \left(\boldsymbol{x}^{(i-1)}, \boldsymbol{y} \right) - \boldsymbol{x}^{(i)} = 0, & 2 \leq i \leq r - 1, \\ \mathcal{R}^{(r)} \left(\boldsymbol{x}^{(r-1)}, \boldsymbol{y} \right) - \boldsymbol{c} = 0, & \end{cases}$$

where $\boldsymbol{y} = (y_1, \ldots, y_n)^\intercal$ and $\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, \ldots, x_n^{(i)} \right)$.

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where $\boldsymbol{y} = (y_1, \ldots, y_n)^\intercal$ and $\boldsymbol{x}^{(i)} = \left(x_1^{(i)}, \ldots, x_n^{(i)}\right)$.

- $\mathcal{F}_{\texttt{GMiMC}_{\text{erf}}} \subset \mathbb{F}_q\left[\boldsymbol{x}^{(i)}, \boldsymbol{y} \;\middle|\; 1 \leq i \leq r-1\right]$ consists of $r \cdot n$ equations in $r \cdot n$ variables.

Transforming The $\mathtt{GMiMC}_{\text{erf}}$ System I

- $\boldsymbol{Y} = \left(\boldsymbol{y}, \boldsymbol{M}_{\mathcal{K}}\boldsymbol{y}, \dots, \boldsymbol{M}_{\mathcal{K}}^{\lceil \frac{r}{n} \rceil - 1}\boldsymbol{y}\right)^{\intercal}$.
- For $1 < i < r$ let us take a closer look:

$$\begin{pmatrix} x_2^{(i-1)} + \left(x_1^{(i-1)} + Y_i + c_i\right)^3 \\ \vdots \\ x_n^{(i-1)} + \left(x_1^{(i-1)} + Y_i + c_i\right)^3 \\ x_1^{(i-1)} \end{pmatrix} = \begin{pmatrix} x_1^{(i)} \\ \vdots \\ x_n^{(i)} \end{pmatrix}.$$

Transforming The $\mathtt{GMiMC}_{\text{erf}}$ System II

- Transform to

$$\begin{pmatrix} x_2^{(i-1)} + \left(x_1^{(i-1)} + Y_i + c_i\right)^3 \\ -x_2^{(i-1)} + x_3^{(i-1)} \\ \vdots \\ -x_2^{(i-1)} + x_n^{(i-1)} \\ x_1^{(i-1)} \end{pmatrix} = \begin{pmatrix} x_1^{(i)} \\ -x_1^{(i)} + x_2^{(i)} \\ \vdots \\ -x_1^{(i)} + x_{n-1}^{(i)} \\ x_n^{(i)} \end{pmatrix}.$$

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- Transform $\mathcal{F}_{\texttt{GMiMC}_{\text{erf}}}$ to r cubic polynomials $\mathcal{F}_{\text{cub}}$ and $r \cdot (n-1)$ affine polynomials $\mathcal{F}_{\text{lin}}$.

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- Transform $\mathcal{F}_{\texttt{GMiMC}_{\text{erf}}}$ to r cubic polynomials $\mathcal{F}_{\text{cub}}$ and $r \cdot (n-1)$ affine polynomials $\mathcal{F}_{\text{lin}}$.
- If $\text{rank}\left(\mathcal{F}_{\text{lin}}\right) = r \cdot (n-1)$, then reduce to r cubic equations in r variables.

Transforming The $\texttt{GMiMC}_{\text{erf}}$ System III

- What happens to the cubic term $\left(x_1^{(i-1)} + Y_i + c_i\right)^3$?
  - $x_1^{(i-1)} + Y_i \mod (\mathcal{F}_{\text{lin}}) = \mathcal{L}_i + a_i$, where $\mathcal{L}_i$ is a linear polynomial and $a_i \in \mathbb{F}_q$.

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- If we can perform a change of variables $\hat{x}_i = \mathcal{L}_i$, then

$$\mathcal{F}_{\text{cub}} \mapsto \left\{ \hat{x}_i^3 + \alpha_i \cdot \hat{x}_i^2 + \mathcal{A}_i = 0, \qquad 1 \leq i \leq r, \right.$$

where $\mathcal{A}_i \in \mathbb{F}_q\left[\hat{x}_i \;\middle|\; 1 \leq i \leq r\right]$ is affine.

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  - Change of variables possible if $\text{rank}\left(\mathcal{L}_1, \ldots, \mathcal{L}_r \right) = r$.
  - For DRL pairwise coprime leading monomials ⇒ DRL Gröbner basis.

Efficient Testing For A DRL Gröbner Basis

- For change of variables we must have that $\text{rank}\left(\mathcal{F}_{\text{lin}}\right) = r \cdot (n - 1)$ and $\text{rank}\left(\mathcal{L}_1, \dots, \mathcal{L}_r\right) = r$.
  - For efficient verification we can delete constant terms in $\mathcal{F}_{\text{lin}}$.
  - Add the linear equations from the cubic terms.

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- E.g., for $1 < i < r$ this yields

$$\left\{ \begin{array}{r} x_1^{(i-1)} + Y_i, \\ \left(-x_2^{(i-1)} + x_{j+1}^{(i-1)}\right) - \left(-x_1^{(i)} + x_j^{(i)}\right), \\ x_1^{(i-1)} - x_n^{(i)} \end{array} \right\}_{2 \leq j \leq n-1} .$$

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- Implement computer algebra program of your choice.

Examples

Example 1

- $q = 2^{64} - 2^{32} + 1$, $n = 3$, $\boldsymbol{M}_{\mathcal{K}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.
- Change of variables works for all integers $r \in [5, 100]$ with $r \not\equiv -1 \mod 6$.

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Example 2

- $q = 2^{31} - 1$, $n = 4$, $\boldsymbol{M}_{\mathcal{K}} = \begin{pmatrix} 5 & 7 & 1 & 3 \\ 4 & 6 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 1 & 4 & 6 \end{pmatrix}$.
- Change of variables works for all integers $r \in [5, 100]$.

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System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis I

$P = \mathbb{F}_q \left[\hat{x}_i \;\middle|\; 1 \leq i \leq r \right]$

- Want to solve the `GMiMC` DRL Gröbner basis for the key

$$\mathcal{G}_{\texttt{GMiMC}_{\text{erf}}} \colon \left\{ \hat{x}_i^3 + \alpha_i \cdot \hat{x}_i^2 + \mathcal{A}_i = 0, \qquad 1 \leq i \leq r. \right.$$

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- As $\mathbb{F}_q$-vector spaces

$$R := P / \left(\mathcal{G}_{\texttt{GMiMC}_{\text{erf}}} \right) \cong P / \left(\hat{x}_i^3 \mid 1 \leq i \leq r \right),$$

$$\dim_{\mathbb{F}_q} (R) = 3^r.$$

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$$\dim_{\mathbb{F}_q} (R) = 3^r.$$

- For every $f \in P$, the multiplication map

$$\theta_f : R \to R, \qquad x \mapsto x \cdot f$$

is $\mathbb{F}_q$-linear $\Rightarrow$ *Multiplication Matrix* $\boldsymbol{M}_f \in \mathbb{F}_q^{3^r \times 3^r}$.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis II

- If $\boldsymbol{a} \in \overline{\mathbb{F}_q}^r$ is root of $(\mathcal{G}_{\texttt{GMiMC}_{\text{erf}}}) \Rightarrow f(\boldsymbol{a})$ is eigenvalue of $\boldsymbol{M}_f$ [KR16, Proposition 6.2.1].

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis II

- If $\boldsymbol{a} \in \overline{\mathbb{F}_q}^r$ is root of $\left(\mathcal{G}_{\texttt{GMiMC}_{\text{erf}}}\right) \Rightarrow f(\boldsymbol{a})$ is eigenvalue of $\boldsymbol{M}_f$ [KR16, Proposition 6.2.1].
- Let $\mathcal{B}' \subset \mathbb{F}_q\left[\hat{x}_i \;\middle|\; 2 \leq i \leq r\right] / \left(\hat{x}_i^3 \;\middle|\; 2 \leq i \leq r\right)$ be a vector space basis, then

$$\mathcal{B} = \mathcal{B}' \cup \hat{x}_1 \cdot \mathcal{B}' \cup \hat{x}_1^2 \cdot \mathcal{B}'$$

is vector space basis of $\mathcal{R}$.

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System Solving With A `GMiMC`$_{\text{erf}}$ Gröbner Basis III

- Then for $f = \hat{x}_1$ [BBL$^+$24, Lemma 2]

$$\boldsymbol{M}_{\hat{x}_1} = \begin{array}{cc} \begin{array}{ccc} \mathcal{B}' & \hat{x}_1 \cdot \mathcal{B}' & \hat{x}_1^2 \cdot \mathcal{B}' \end{array} & \\ \begin{pmatrix} 0 & \boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{I} \\ \boldsymbol{M}_0 & \boldsymbol{M}_1 & \boldsymbol{M}_2 \end{pmatrix} & \begin{array}{c} \hat{x}_1 \cdot \mathcal{B}' \\ \hat{x}_1^2 \cdot \mathcal{B}' \\ \hat{x}_1^3 \cdot \mathcal{B}' \end{array} \end{array},$$

$$\begin{aligned} \chi_{\hat{x}_1}(x) &= \det \left(\boldsymbol{I} \cdot x - \boldsymbol{M}_{\hat{x}_1} \right) \\ &= \pm \det \left(\boldsymbol{I} \cdot x^3 - \boldsymbol{M}_2 \cdot x^2 - \boldsymbol{M}_1 \cdot x - \boldsymbol{M}_0 \right). \end{aligned}$$

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis III

- Then for $f = \hat{x}_1$ [BBL$^+$24, Lemma 2]

$$\begin{array}{cc} & \begin{array}{ccc} \mathcal{B}' & \hat{x}_1 \cdot \mathcal{B}' & \hat{x}_1^2 \cdot \mathcal{B}' \end{array} & \\ \boldsymbol{M}_{\hat{x}_1} = & \begin{pmatrix} 0 & \boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{I} \\ \boldsymbol{M}_0 & \boldsymbol{M}_1 & \boldsymbol{M}_2 \end{pmatrix} & \begin{array}{c} \hat{x}_1 \cdot \mathcal{B}' \\ \hat{x}_1^2 \cdot \mathcal{B}' \\ \hat{x}_1^3 \cdot \mathcal{B}' \end{array} \end{array},$$

$$\begin{aligned} \chi_{\hat{x}_1}(x) &= \det\left(\boldsymbol{I} \cdot x - \boldsymbol{M}_{\hat{x}_1}\right) \\ &= \pm \det\left(\boldsymbol{I} \cdot x^3 - \boldsymbol{M}_2 \cdot x^2 - \boldsymbol{M}_1 \cdot x - \boldsymbol{M}_0\right). \end{aligned}$$

- Characteristic polynomial can be computed in [BBL$^+$24, §3.2]

$$\mathcal{C}_{\text{char-poly}} = \mathcal{O}\left(3 \cdot \log_2(3)^2 \cdot \left(1 + \log_2\left(\log_2(3)\right)\right) \cdot 3^{\omega \cdot (r-1)}\right)$$

field operations, where $2 \leq \omega < 2.371552$ [WXXZ24].

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis IV

$\texttt{GMiMC}_{\text{erf}}$ Gröbner basis

$$\begin{aligned}
\hat{x}_1^3 + \alpha_1 \cdot \hat{x}_1^2 + \mathcal{A}_1 &= 0, \\
\hat{x}_2^3 + \alpha_2 \cdot \hat{x}_2^2 + \mathcal{A}_2 &= 0, \\
&\vdots \\
\hat{x}_n^3 + \alpha_n \cdot \hat{x}_n^2 + \mathcal{A}_n &= 0.
\end{aligned}$$

$\texttt{GMiMC}_{\text{erf}}$ substitution variables

$$\begin{aligned}
\hat{x}_1 &= y_1, \\
\hat{x}_2 &= x_1^{(1)} + y_2, \\
&\vdots \\
\hat{x}_n &= x_1^{(n-1)} + y_n,
\end{aligned}$$

- Characteristic polynomial for $\hat{x}_1$ $\Rightarrow$ Key guess for y_1.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis IV

$\texttt{GMiMC}_{\text{erf}}$ Gröbner basis

$$\begin{gathered}\hat{x}_1^3 + \alpha_1 \cdot \hat{x}_1^2 + \mathcal{A}_1 = 0,\\ \hat{x}_2^3 + \alpha_2 \cdot \hat{x}_2^2 + \mathcal{A}_2 = 0,\\ \vdots \\ \hat{x}_n^3 + \alpha_n \cdot \hat{x}_n^2 + \mathcal{A}_n = 0.\end{gathered}$$

$\texttt{GMiMC}_{\text{erf}}$ substitution variables

$$\begin{aligned}\hat{x}_1 &= y_1,\\ \hat{x}_2 &= x_1^{(1)} + y_2,\\ &\vdots\\ \hat{x}_n &= x_1^{(n-1)} + y_n,\end{aligned}$$

- Characteristic polynomial for $\hat{x}_1 \Rightarrow$ Key guess for y_1.
- Substitute $\hat{x}_1 = y_1$ back into system to eliminate $\hat{x}_1$.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis IV

$\texttt{GMiMC}_{\text{erf}}$ Gröbner basis

$$\begin{aligned} \hat{x}_1^3 + \alpha_1 \cdot \hat{x}_1^2 + \mathcal{A}_1 &= 0, \\ \hat{x}_2^3 + \alpha_2 \cdot \hat{x}_2^2 + \mathcal{A}_2 &= 0, \\ &\vdots \\ \hat{x}_n^3 + \alpha_n \cdot \hat{x}_n^2 + \mathcal{A}_n &= 0. \end{aligned}$$

$\texttt{GMiMC}_{\text{erf}}$ substitution variables

$$\begin{aligned} \hat{x}_1 &= y_1, \\ \hat{x}_2 &= x_1^{(1)} + y_2, \\ &\vdots \\ \hat{x}_n &= x_1^{(n-1)} + y_n, \end{aligned}$$

- Characteristic polynomial for $\hat{x}_1 \Rightarrow$ Key guess for y_1.
- Substitute $\hat{x}_1 = y_1$ back into system to eliminate $\hat{x}_1$.
- Ignore first polynomial $\Rightarrow$ DRL Gröbner basis in $r - 1$ variables.

System Solving With A $\texttt{GMiMC}_{\mathrm{erf}}$ Gröbner Basis IV

$\texttt{GMiMC}_{\mathrm{erf}}$ Gröbner basis

$$\begin{aligned} \hat{x}_1^3 + \alpha_1 \cdot \hat{x}_1^2 + \mathcal{A}_1 &= 0, \\ \hat{x}_2^3 + \alpha_2 \cdot \hat{x}_2^2 + \mathcal{A}_2 &= 0, \\ &\vdots \\ \hat{x}_n^3 + \alpha_n \cdot \hat{x}_n^2 + \mathcal{A}_n &= 0. \end{aligned}$$

$\texttt{GMiMC}_{\mathrm{erf}}$ substitution variables

$$\begin{aligned} \hat{x}_1 &= y_1, \\ \hat{x}_2 &= x_1^{(1)} + y_2, \\ &\vdots \\ \hat{x}_n &= x_1^{(n-1)} + y_n, \end{aligned}$$

- Characteristic polynomial for $\hat{x}_1$ ⇒ Key guess for y_1.
- Substitute $\hat{x}_1 = y_1$ back into system to eliminate $\hat{x}_1$.
- Ignore first polynomial ⇒ DRL Gröbner basis in $r - 1$ variables.
- Characteristic polynomial for $\hat{x}_2$ ⇒ Key guess for y_2.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis IV

$\texttt{GMiMC}_{\text{erf}}$ Gröbner basis

$$\begin{aligned}
\hat{x}_1^3 + \alpha_1 \cdot \hat{x}_1^2 + \mathcal{A}_1 &= 0, \\
\hat{x}_2^3 + \alpha_2 \cdot \hat{x}_2^2 + \mathcal{A}_2 &= 0, \\
&\vdots \\
\hat{x}_n^3 + \alpha_n \cdot \hat{x}_n^2 + \mathcal{A}_n &= 0.
\end{aligned}$$

$\texttt{GMiMC}_{\text{erf}}$ substitution variables

$$\begin{aligned}
\hat{x}_1 &= y_1, \\
\hat{x}_2 &= x_1^{(1)} + y_2, \\
&\vdots \\
\hat{x}_n &= x_1^{(n-1)} + y_n,
\end{aligned}$$

- Characteristic polynomial for $\hat{x}_1$ ⇒ Key guess for y_1.
- Substitute $\hat{x}_1 = y_1$ back into system to eliminate $\hat{x}_1$.
- Ignore first polynomial ⇒ DRL Gröbner basis in $r - 1$ variables.
- Characteristic polynomial for $\hat{x}_2$ ⇒ Key guess for y_2.
- Iterate.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis V

- Dominating complexity for iterated solving

$$
\begin{aligned}
&\mathcal{O} \left(\sum_{i=1}^{n} N^{i-1} \cdot \mathcal{C}_{\text{char-poly}}(r - i + 1, \omega) \right) \\
&\in \mathcal{O} \left(3 \cdot \log_2(3)^2 \cdot \Big(1 + \log_2 \big(\log_2(3) \big) \Big) \cdot \frac{3^{\omega \cdot r} - N^n}{3^{\omega} - N} \right),
\end{aligned}
$$

where N is a bound on the $\mathbb{F}_q$-valued solutions in each iteration.

System Solving With A $\texttt{GMiMC}_{\mathrm{erf}}$ Gröbner Basis V

- Dominating complexity for iterated solving

$$\mathcal{O}\left(\sum_{i=1}^{n} N^{i-1} \cdot \mathcal{C}_{\text{char-poly}}(r - i + 1, \omega)\right)$$
$$\in \mathcal{O}\left(3 \cdot \log_2(3)^2 \cdot \Big(1 + \log_2\big(\log_2(3)\big)\Big) \cdot \frac{3^{\omega \cdot r} - N^n}{3^\omega - N}\right),$$

  where N is a bound on the $\mathbb{F}_q$-valued solutions in each iteration.

- N a priori unknown but can be avoided with additional plain/ciphertext samples.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis V

- Dominating complexity for iterated solving

$$\mathcal{O}\left(\sum_{i=1}^{n} N^{i-1} \cdot \mathcal{C}_{\text{char-poly}}(r - i + 1, \omega)\right)$$
$$\in \mathcal{O}\left(3 \cdot \log_2(3)^2 \cdot \Big(1 + \log_2\big(\log_2(3)\big)\Big) \cdot \frac{3^{\omega \cdot r} - N^n}{3^\omega - N}\right),$$

  where N is a bound on the $\mathbb{F}_q$-valued solutions in each iteration.
- N a priori unknown but can be avoided with additional plain/ciphertext samples.
  - Compute DRL Gröbner bases in parallel.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis V

- Dominating complexity for iterated solving

$$\mathcal{O} \left(\sum_{i=1}^{n} N^{i-1} \cdot \mathcal{C}_{\text{char-poly}}(r - i + 1, \omega) \right)$$
$$\in \mathcal{O} \left(3 \cdot \log_2(3)^2 \cdot \Big(1 + \log_2 \big(\log_2(3) \big) \Big) \cdot \frac{3^{\omega \cdot r} - N^n}{3^{\omega} - N} \right),$$

  where N is a bound on the $\mathbb{F}_q$-valued solutions in each iteration.

- N a priori unknown but can be avoided with additional plain/ciphertext samples.
  - Compute DRL Gröbner bases in parallel.
  - Compute characteristic polynomials in parallel.

System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis V

- Dominating complexity for iterated solving

$$\mathcal{O} \left(\sum_{i=1}^{n} N^{i-1} \cdot \mathcal{C}_{\text{char-poly}}(r - i + 1, \omega) \right)$$
$$\in \mathcal{O} \left(3 \cdot \log_2(3)^2 \cdot \Big(1 + \log_2 \big(\log_2(3) \big) \Big) \cdot \frac{3^{\omega \cdot r} - N^n}{3^\omega - N} \right),$$

  where N is a bound on the $\mathbb{F}_q$-valued solutions in each iteration.

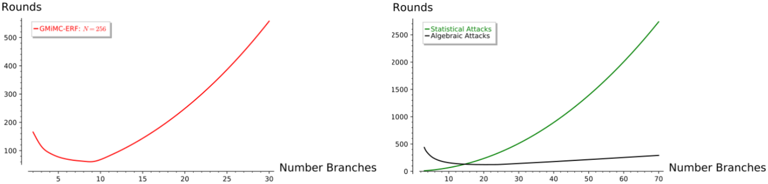
- N a priori unknown but can be avoided with additional plain/ciphertext samples.
  - Compute DRL Gröbner bases in parallel.
  - Compute characteristic polynomials in parallel.
  - Filter key guesses via GCD.

Outline

1. Motivation
2. Gröbner Bases
3. $\texttt{GMiMC}_{\text{erf}}$
4. Constructing A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis
   - Examples
5. System Solving With A $\texttt{GMiMC}_{\text{erf}}$ Gröbner Basis
6. $\texttt{GMiMC}_{\text{erf}}$ Cryptanalysis
   - What went wrong?
7. Further Outlook
8. References

GMiMC$_{\text{erf}}$ Cryptanalysis I

- GMiMC team claimed there is trade-off between algebraic and statistical cryptanalysis.



(a) $\kappa = 256$.

(b) $\kappa = 1024$.

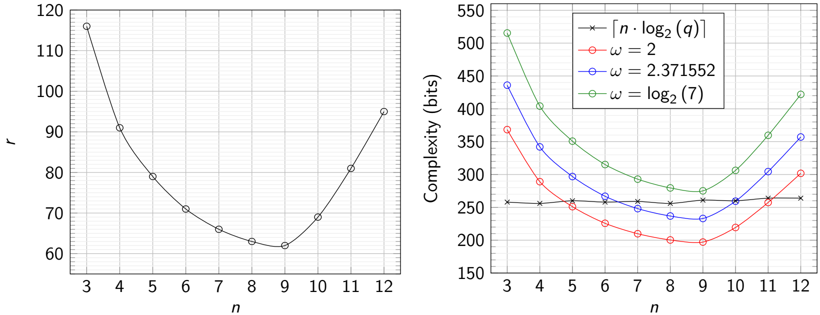
Figure: Figrues from [AGP$^+$19b, Fig. 5, 7].

$\texttt{GMiMC}_{\text{erf}}$ Cryptanalysis II

- For Gröbner basis cryptanalysis of $\texttt{GMiMC}_{\text{erf}}$:
  - Fix security level κ.
  - Increase $n \in \mathbb{Z}_{\geq 0}$ and set $\log_2(q) = \left\lceil \frac{\kappa}{n} \right\rceil$.
  - Multivariate key $\boldsymbol{k} \in \mathbb{F}_q^n$.
  - Compute round number r for (q, n, κ) for $\texttt{GMiMC}_{\text{erf}}$ with round numbers tool.[1]
  - Assume that we can construct DRL Gröbner basis via substitution.
  - Use $N = 1$ in $\mathcal{O}\left(3 \cdot \log_2(3)^2 \cdot \Big(1 + \log_2\big(\log_2(3)\big)\Big) \cdot \frac{3^{\omega \cdot r} - N^n}{3^\omega - N}\right)$.

[1] https://extgit.iaik.tugraz.at/krypto/gmimc-implementation

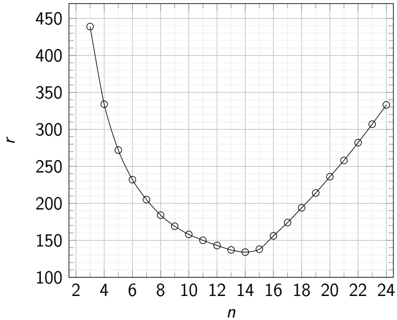
$\mathtt{GMiMC}_{\text{erf}}$, $\kappa = 256$



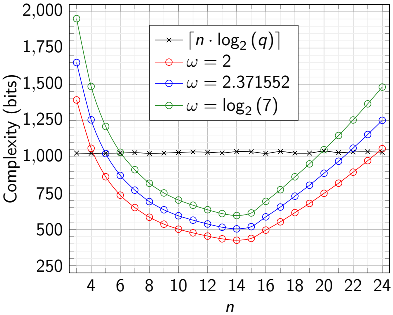
(a) Round numbers, $\kappa = 256$.

(b) Complexities, $\kappa = 256$.

$\texttt{GMiMC}_{\text{erf}}$, $\kappa = 1024$

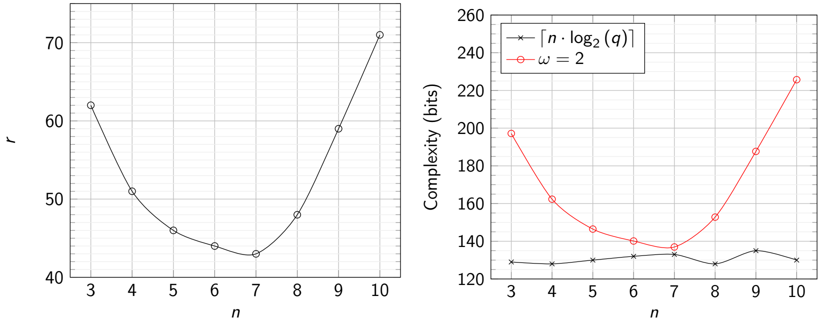


(a) Round numbers, $\kappa = 1024$.



(b) Complexities, $\kappa = 1024$.

$\texttt{GMiMC}_{\text{erf}}$, $\kappa = 128$



(a) Round numbers, $\kappa = 128$.

(b) Complexities, $\kappa = 128$.

What went wrong? I

- `GMiMC`$_{\text{erf}}$ team modeled `GMiMC`$_{\text{erf}}$ as n polynomials in n key variables of degrees $\leq 3^r$.
    - Complexity under regularity assumption: $\approx \mathcal{O}\left(\binom{n+r^{r-n}}{3^{r-n}}^{\omega}\right)$.

What went wrong? I

- $\texttt{GMiMC}_{\text{erf}}$ team modeled $\texttt{GMiMC}_{\text{erf}}$ as n polynomials in n key variables of degrees $\leq 3^r$.
    - Complexity under regularity assumption: $\approx \mathcal{O}\left(\binom{n+r^{r-n}}{3^{r-n}}^{\omega}\right)$.
- Computing $\texttt{GMiMC}_{\text{erf}}$ Gröbner basis for r cubic polynomials.
    - Complexity under regularity assumption: $\mathcal{O}\left(\binom{3 \cdot r+1}{2 \cdot r+1}^{\omega}\right)$.

What went wrong? I

- $\texttt{GMiMC}_{\mathsf{erf}}$ team modeled $\texttt{GMiMC}_{\mathsf{erf}}$ as n polynomials in n key variables of degrees $\leq 3^r$.
  - Complexity under regularity assumption: $\approx \mathcal{O}\left(\binom{n+r^{r-n}}{3^{r-n}}^{\omega}\right)$.
- Computing $\texttt{GMiMC}_{\mathsf{erf}}$ Gröbner basis for r cubic polynomials.
  - Complexity under regularity assumption: $\mathcal{O}\left(\binom{3 \cdot r+1}{2 \cdot r+1}^{\omega}\right)$.
- Solving $\texttt{GMiMC}_{\mathsf{erf}}$ system after change of variables.
  - Complexity: $\approx \mathcal{O}\left(3^{\omega \cdot r}\right)$.

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What went wrong? II

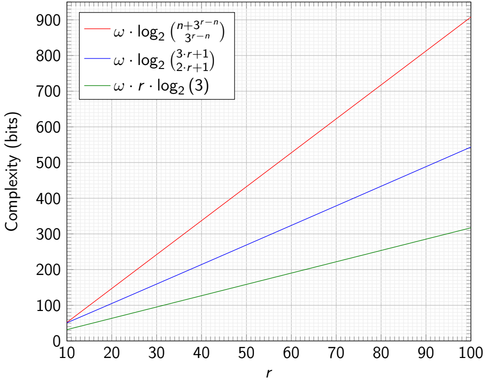

Figure: $\texttt{GMiMC}_{\text{erf}}$ complexity gap with $n = 3$ and $\omega = 2$.

Further Outlook

- "Variable elimination + change of coordinates ⇒ Gröbner basis" can be applied to other Feistel designs too.
    - Other members of `GMiMC` family (`GMiMCHash`, $\texttt{GMiMC}_{\mathsf{crf}}$).
    - Hydra [Ste24].
- Can be used at design stage to have Gröbner basis from the start.
- When is the `GMiMC` analysis going to be public?
    - In my PhD thesis: soon.
    - In a paper: later.

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